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Optimal replenishment policies with allowable shortages for a product life cycle

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Abstract

In this paper, we investigate replenishment policies with allowable shortages by considering a general, time-varying, continuous, and deterministic demand function for a product life cycle. The objective is to optimally determine the number of inventory replenishments, the inventory replenishment time points, and the beginning time points of shortages within the product life cycle by minimizing the total relevant costs of the inventory replenishment system. The proposed problem is mathematically formulated as a mixed-integer nonlinear programming model. A complete search procedure is developed to find the optimal solution by employing the properties derived in this paper and the well-known Nelder–Mead algorithm. Also, several numerical examples and the corresponding sensitivity analyses are carried out to illustrate the features of our model by utilizing the search procedure developed in this paper.

Keywords: Shortage; Replenishment; Time-varying demand; Nelder–Mead algorithm

1. Introduction

This paper investigates replenishment policies with allowable shortages by considering a general, time-varying, continuous, and deterministic demand function for a product life cycle. The objective is to optimally determine the number of inventory replenishments, the inventory replenishment time points, and the beginning time points of shortages within the product life cycle by minimizing the total relevant costs of the inventory replenishment system. A complete search procedure is developed to find the optimal solution by employing the properties derived in this paper and the well-known Nelder–Mead algorithm.

One of the underlying assumptions for the classical economic order quantity (EOQ)/economic manufacturing quantity (EMQ) model is that the planning horizon is assumed to be infinity. Schwarz [1] relaxed the assumption of the infinite planning horizon and presented a production inventory model to determine the optimal policies for the economic manufacturing quantity problem under the condition that the planning horizon is finite. In contrast to the constant demand considered in Schwarz [1], Donaldson [2] examined the case of a linear trend in demand and provided

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a computationally simple procedure to determine the optimal inventory replenishment time points in a finite planning horizon. After Donaldson [2], numerous research works have been carried out by incorporating linear-increasing demand into inventory models under a variety of circumstances (see e.g., Silver [3], Henery [4], Ritchie [5], Amrani and Rand [6], Yang and Rand [7], Teng [8], Goyal [9] and Teng [10]). The literature concerning the replenishment problem for the case of linear-decreasing demand is also well documented (see e.g., Brosseau [11], Hariga [12], Lo et al. [13], Zhao et al. [14], and Goyal and Giri [15]). In contrast to the case of linear trend demand, there have been several researchers who have made contributions to the case of non-linear increasing demand for the replenishment problem (see e.g., Yang et al. [16], Wang [17], and Yang et al. [18]). Roger [19] investigated the replenishment policies for the life cycle of a product which has a continuous, time-varying deterministic demand pattern. The solution procedure in Roger [19] is that the demand over the life cycle is classified into several segments and one solves the replenishment problem within each segment without considering the product life cycle as a whole. Another extension of the inventory model with time-varying demands is to consider the case where shortages are allowed (see e.g., Benkherouf [20], Dave [21], Dave [22], Deb and Chaudhuri [23], Goyal [24], Hariga [25], Murdeshwar [26] and Teng et al. [27]).

In this paper, we investigate the replenishment policies with allowable shortages by considering a general, time-varying, continuous, and deterministic demand function for a product life cycle. The objective is to optimally determine the number of inventory replenishments, the inventory replenishment time points, and the beginning time points of shortages within the product life cycle by minimizing the total relevant costs of the inventory replenishment system. By employing the properties derived in this paper and the well-known Nelder–Mead algorithm, a complete search procedure is developed to find the optimal solution. In Section 2, some basic assumptions and mathematical notations are presented. In Section 3, we formulate the proposed problem as a cost minimization problem, where the number of inventory replenishments, the inventory replenishment time points, and the beginning time points of shortages within the product life cycle are the decision variables. Following the mathematical formulation, in Section 4, a complete search procedure is provided to find the optimal solution by employing the Nelder–Mead algorithm. In Section 5, several numerical examples and the corresponding sensitivity analyses are carried out to illustrate the features of our model by utilizing the search procedure developed in Section 4. Finally, some concluding remarks are made in Section 6.

2. Assumptions and notations

In order to construct and analyze the mathematical model discussed in this paper, we introduce some assumptions and notations as follows.

2.1. Assumptions

1. A single product is considered in this paper.
2. The planning horizon under consideration is assumed to be finite.
3. Shortages are allowed and are assumed to be completely backordered.
4. Replenishments occur instantaneously at an infinite rate.
5. The initial inventory level at the beginning time of the planning horizon is assumed to be zero.
6. The inventory level depletes to zero at the end of the planning horizon.
7. The time vale of money (i.e., discounting) is ignored.
8. The deteriorating nature of products is not considered.

In addition, we assume that the demand function of the product in the market is a function of time, which follows the shape of a product-life-cycle. The demand function with a product-life-cycle shape can be classified into 4 stages. At the beginning of the planning horizon, the demand increases very slowly. As time goes on, the demand increases very rapidly and eventually reaches a peak. At the last stage, the demand decreases with time and reaches zero at the end of the planning horizon. Specifically, we assume that the demand function is a revised version of the Beta distribution function. Namely,

$$f(t) = \frac{Q}{B(\alpha, \beta)} t^{\alpha-1}(H-t)^{\beta-1} \tag{1}$$

$$B(\alpha, \beta) = \int_0^H t^{\alpha-1}(H-t)^{\beta-1}\mathrm{d}t \tag{2}$$

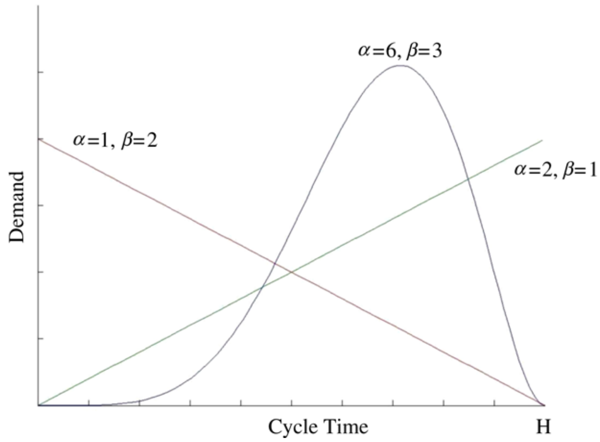

Fig. 1. Different sets of parameters α and β for the Beta distribution demand function.

where H is the planning horizon under consideration, Q is the cumulative quantity of the demand in the planning horizon, and α, β are the constant parameters for the revised Beta distribution function. Different values of α and β will give different shapes of the demand function. In Fig. 1, we show three different sets of parameters for α and β to describe the demand function. If $\alpha = 1$ and $\beta = 2$, the demand is expressed as a linear decreasing function. On the other hand, if $\alpha = 2$ and $\beta = 1$, the demand is expressed as a linear increasing function. If $\alpha = 6$ and $\beta = 3$, the demand function forms up in a product-life-cycle shape.

2.2. Notations

W: The total relevant costs including unit purchasing cost, fixed replenishment cost, inventory holding cost, and shortage cost
H: The planning horizon under consideration
C_0: Unit purchasing cost per item
C_1: The fixed replenishment cost per replenishment
C_2: The inventory holding cost per unit per unit time
C_3: The shortage cost per unit per unit time
Q: The cumulative quantity of the demand in the planning horizon
n: The number of replenishments over $[0, H]$ (a decision variable)
t_i: The ith replenishment time point, $i = 0, 1, 2, \ldots, n$, with $t_0 = 0$ and $t_n = H$ (decision variables)
s_i: The ith beginning time point of shortage, $i = 1, 2, \ldots, n$, with $s_n = H$ (decision variables)
α, β: The constant parameters for the revised Beta distribution demand function
$f(t)$ the demand function at time t, and $0 \le t \le H$.

3. Mathematical formulation of the model

The total relevant costs considered in our model include the fixed replenishment cost, the unit purchasing cost, the inventory holding cost, and the inventory shortage cost. The mathematical formulation of the objective function for our model can be expressed in Eq. (3).

$$W(n, \{t_i\}, \{s_i\}) = C_0 \int_0^H f(t)\mathrm{d}t + nC_1 + C_2 \sum_{i=0}^{n-1} \int_{t_i}^{s_{i+1}} (t - t_i) f(t)\mathrm{d}t + C_3 \sum_{i=0}^{n-1} \int_{s_{i+1}}^{t_{i+1}} (t_{i+1} - t) f(t)\mathrm{d}t. \quad (3)$$

The first term in Eq. (3) is the total purchasing costs for the product while the second term is the total replenishment costs in the planning horizon. The third term and the fourth term of Eq. (3) represent the total inventory holding costs and the total shortage costs in the planning horizon, respectively. Moreover, we note that the decision variables in expression (3) includes an integer value n, $(n - 1)$ real values of t_i, and $(n - 1)$ real values of s_i.

If we take the first derivative of $W(n, \{t_i\}, \{s_i\})$ with respect to s_i, then we can have the following result.

$$\frac{\partial W(n, \{t_i\}, \{s_i\})}{\partial s_i} = C_2(s_i - t_{i-1}) f(s_i) - C_3(t_i - s_i) f(s_i) = 0. \tag{4}$$

By rearranging Eq. (4), s_i can be expressed by t_i and t_{i-1}, which is shown as in Eq. (5).

$$s_i = \frac{C_2 t_{i-1} + C_3 t_i}{C_2 + C_3} = \frac{C_3}{C_2 + C_3}(t_i - t_{i-1}) + t_{i-1}. \tag{5}$$

Moreover, the second partial derivative of $W(n, \{t_i\}, \{s_i\})$ with respect to s_i can be easily shown to be positive by evaluating s_i at which Eq. (4) is satisfied. Therefore, we can substitute s_i by $\frac{C_3}{C_2+C_3}(t_i - t_{i-1}) + t_{i-1}$ into the objective function. Hence, the objective function can be simplified as follows.

$$\begin{aligned} W(n, \{t_i\}) = {} & C_0 \int_0^H f(t)\mathrm{d}t + nC_1 + C_2 \sum_{i=0}^{n-1} \int_{t_i}^{\frac{C_3}{C_2+C_3}(t_{i+1}-t_i)+t_i} (t - t_i) f(t)\mathrm{d}t \\ & + C_3 \sum_{i=0}^{n-1} \int_{\frac{C_3}{C_2+C_3}(t_{i+1}-t_i)+t_i}^{t_{i+1}} (t_{i+1} - t) f(t)\mathrm{d}t. \end{aligned} \tag{6}$$

We note that the decision variables in the objective function (6) are the discrete integer variable n and positive real variables of t_i for $i = 1, 2, \ldots, n-1$. If we obtain optimal t_i^*, then s_i^* can be obtained by Eq. (5) accordingly. Hence, the problem discussed in this paper becomes a cost minimization problem with a mixed-integer multidimensional objective function. It is almost impossible to have analytical closed form solutions for the proposed model. In the following section, we will present a complete search procedure to find out the optimal number of replenishments n^* and the corresponding optimal replenishment time point t_i^* and the optimal beginning time points of shortage s_i^* for $i = 1, 2, \ldots, n-1$. Specifically, for given any n, we will utilize the Nelder–Mead algorithm to solve for the optimal t_i^*.

4. Solution procedure

In this section, we will first briefly review the Nelder–Mead algorithm [28]. Next, we will show how to apply the Nelder–Mead algorithm to find the optimal t_i^* and s_i^* under the condition that n is given as a fixed integer. Then, the algorithm to find the optimal n^* is presented. A complete search procedure to determine the optimal n^*, t_i^*, and s_i^* is shown in a flow chart.

4.1. Review of the Nelder–Mead algorithm

The Nelder–Mead algorithm was proposed as a derivative-free method for minimizing a real-valued function $f(\mathbf{x})$ for $\mathbf{x} \in R^n$. The essential concept behind the search procedure of the Nelder–Mead algorithm is that the worst vertex is replaced by a new better vertex at each iteration until the vertices converge. Additionally, every iteration of the Nelder–Mead algorithm has an interesting geometrical interpretation. We summarize the search procedure of the Nelder–Mead algorithm as follows.

1. Before performing the algorithm, four scalar parameters must be specified to define a complete Nelder–Mead method which are the coefficients of reflection (ω), expansion (θ), contraction (γ), and shrinkage (σ). Lagarias et al. [29] indicates that these four coefficients should satisfy

$$\omega > 0, \qquad \theta > 1, \qquad \theta > \omega, \qquad 1 > \gamma > 0, \qquad 1 > \sigma > 0 \tag{7}$$

and the nearly universal choices used in the standard of the Nelder–Mead algorithm which should be

$$\omega = 1, \qquad \theta = 2, \qquad \gamma = \frac{1}{2}, \qquad \sigma = \frac{1}{2}. \tag{8}$$

2. At the beginning of the each iteration, $(n+1)$ vertices are identified, each of which is a point in R^n. Assuming that each iteration begins by ordering and labeling these vertices as $\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^{n+1}$, such that $f(\mathbf{x}^1) \leq f(\mathbf{x}^2) \leq \cdots \leq f(\mathbf{x}^{n+1})$. Define $\bar{\mathbf{x}} = \frac{\sum_{i=1}^{n} \mathbf{x}^i}{n}$ as the centroid of n best points (all vertices except for $\mathbf{x}^{n+1}$).
3. The result of any iteration is either (1) a single new vertex, which replaces $\mathbf{x}^{n+1}$ in the set of vertices for next iteration, or (2) a shrink, where a set of n new points together with $\mathbf{x}^1$ form up the new set of vertices for the next iteration. Specifically, four different operations, which are REFLECTION, EXPANSION, INSIDE CONTRACTION, and OUTSIDE CONTRACTION, yield a single new vertex while the SHRINK operation replaces all but one (i.e., $\mathbf{x}^1$) vertices to form the new set of vertices.
4. The REFLECTION operation means the worst point $\mathbf{x}^{n+1}$ is reflected through the centroid $\bar{\mathbf{x}}$ by a factor of ω. Based on the result of the REFLECTION operation, the EXPANSION operation extends the reflection point along with the reflection path by a factor of θ.
5. The OUTSIDE CONTRACTION operation moves the reflection point towards the centroid by a factor of γ. On the other hand, the INSIDE CONTRACTION operation moves the worse point towards centroid $\bar{\mathbf{x}}$ by a factor of γ.
6. The SHRINK operation causes all vertices but the best point to have their distances to the best point reduced by a factor of σ.

For those five different types of operations in the Nelder–Mead algorithm, please refer to Fig. 2 in details for geometric interpretations.

4.2. Algorithmic structure for the Nelder–Mead algorithm

First we suppose that we have m variables to solve. An iteration of the Nelder–Mead algorithm can be described as follows.

1. Given parameters $\omega, \theta, \gamma, \sigma$, order the $(m+1)$ vertices to satisfy $f(\mathbf{x}^1) \leq f(\mathbf{x}^2) \leq \cdots \leq f(\mathbf{x}^m) \leq f(\mathbf{x}^{m+1})$. Define $\bar{\mathbf{x}} = \frac{\sum_{i=1}^{m} \mathbf{x}^i}{m}$ is the centroid of m best points (all vertices except for $\mathbf{x}^{m+1}$).
2. Calculate $\mathbf{x}^r$ from $\mathbf{x}^r = \bar{\mathbf{x}} + \omega(\bar{\mathbf{x}} - \mathbf{x}^{m+1}) = (1+\omega) \times \bar{\mathbf{x}} - \omega \times \mathbf{x}^{m+1}$ and evaluate $f(\mathbf{x}^r)$. If $f(\mathbf{x}^1) \leq f(\mathbf{x}^r) < f(\mathbf{x}^m)$, then replace $\mathbf{x}^{m+1}$ with the reflection vertex $\mathbf{x}^r$ and terminate the iteration.
3. If $f(\mathbf{x}^r) < f(\mathbf{x}^1)$, calculate the expansion vertex $\mathbf{x}^e$ from $\mathbf{x}^e = \bar{\mathbf{x}} + \theta(\mathbf{x}^r - \bar{\mathbf{x}}) = (1+\omega\theta)\bar{\mathbf{x}} - \omega\theta\mathbf{x}^{m+1}$ and evaluate $f(\mathbf{x}^e)$. If $f(\mathbf{x}^e) < f(\mathbf{x}^r)$, then replace $\mathbf{x}^{m+1}$ with the expansion vertex $\mathbf{x}^e$ and terminate the iteration; otherwise, replace $\mathbf{x}^{m+1}$ with the reflection vertex $\mathbf{x}^r$ and terminate the iteration.
4. If $f(\mathbf{x}^m) \leq f(\mathbf{x}^r) < f(\mathbf{x}^{m+1})$, then calculate $\mathbf{x}^{\mathrm{co}}$ from $\mathbf{x}^{\mathrm{co}} = \bar{\mathbf{x}} + \gamma(\mathbf{x}^r - \bar{\mathbf{x}}) = (1+\omega\gamma)\bar{\mathbf{x}} - \omega\gamma\mathbf{x}^{m+1}$ and evaluate $f(\mathbf{x}^{\mathrm{co}})$. If $f(\mathbf{x}^{\mathrm{co}}) \leq f(\mathbf{x}^r)$, then replace $\mathbf{x}^{m+1}$ with the outside contraction vertex $\mathbf{x}^{\mathrm{co}}$ and terminate the iteration; otherwise, go to step 6 to perform a shrink.
5. On the other hand, if $f(\mathbf{x}^{m+1}) \leq f(\mathbf{x}^r)$, then calculate $\mathbf{x}^{\mathrm{ci}}$ from $\mathbf{x}^{\mathrm{ci}} = \bar{\mathbf{x}} - \gamma(\bar{\mathbf{x}} - \mathbf{x}^{m+1}) = (1-\gamma)\bar{\mathbf{x}} + \gamma\mathbf{x}^{m+1}$ and evaluate $f(\mathbf{x}^{\mathrm{ci}})$. If $f(\mathbf{x}^{\mathrm{ci}}) < f(\mathbf{x}^{m+1})$, then replace $\mathbf{x}^{m+1}$ with the inside contraction vertex $\mathbf{x}^{\mathrm{ci}}$ and terminate the iteration; otherwise, go to step 6 to perform a shrink.
6. Perform a shrink operation. Calculate $\mathbf{v}^i = \mathbf{x}^1 + \sigma(\mathbf{x}^i - \mathbf{x}^1)$ for $i = 2, 3, \ldots, m+1$. Replace $\mathbf{x}^2$ by $\mathbf{v}^2$, $\mathbf{x}^3$ by $\mathbf{v}^3, \ldots, \mathbf{x}^m$ by $\mathbf{v}^m$ and $\mathbf{x}^{m+1}$ by $\mathbf{v}^{m+1}$. That is, the new set of vertices is $\mathbf{x}^1, \mathbf{v}^2, \mathbf{v}^3, \ldots, \mathbf{v}^m, \mathbf{v}^{m+1}$.

The flow chart of iterations in the Nelder–Mead method can be depicted as Fig. 3.

4.3. Applying the Nelder–Mead algorithm to solve our problem

In the previous subsection, we briefly narrate the principle and the process of the Nelder–Mead algorithm. In this subsection, we will show how to apply the Nelder–Mead algorithm to solve our problem.

Referring to Eq. (6), we seek to minimize the total cost through determining the optimal number of replenishments in the planning horizon n^*, and the corresponding replenishment time point t_i^* for $i = 1, 2, \ldots, n^* - 1$. Once we obtain the optimal replenishment time point t_i^*, the optimal s_i^* can be obtained by Eq. (5) accordingly. Therefore, we can separate our problem into two parts: (1) given any n, finding the optimal replenishments in the planning time points t_i^* and shortages in the planning time points s_i^* for $i = 1, 2, \ldots, n-1$; and (2) finding the optimal number of replenishments in the planning horizon n^*. We will first describe how to find the optimal replenishment time points t_i^* by employing the Nelder–Mead algorithm under the condition that the number of replenishments in the planning

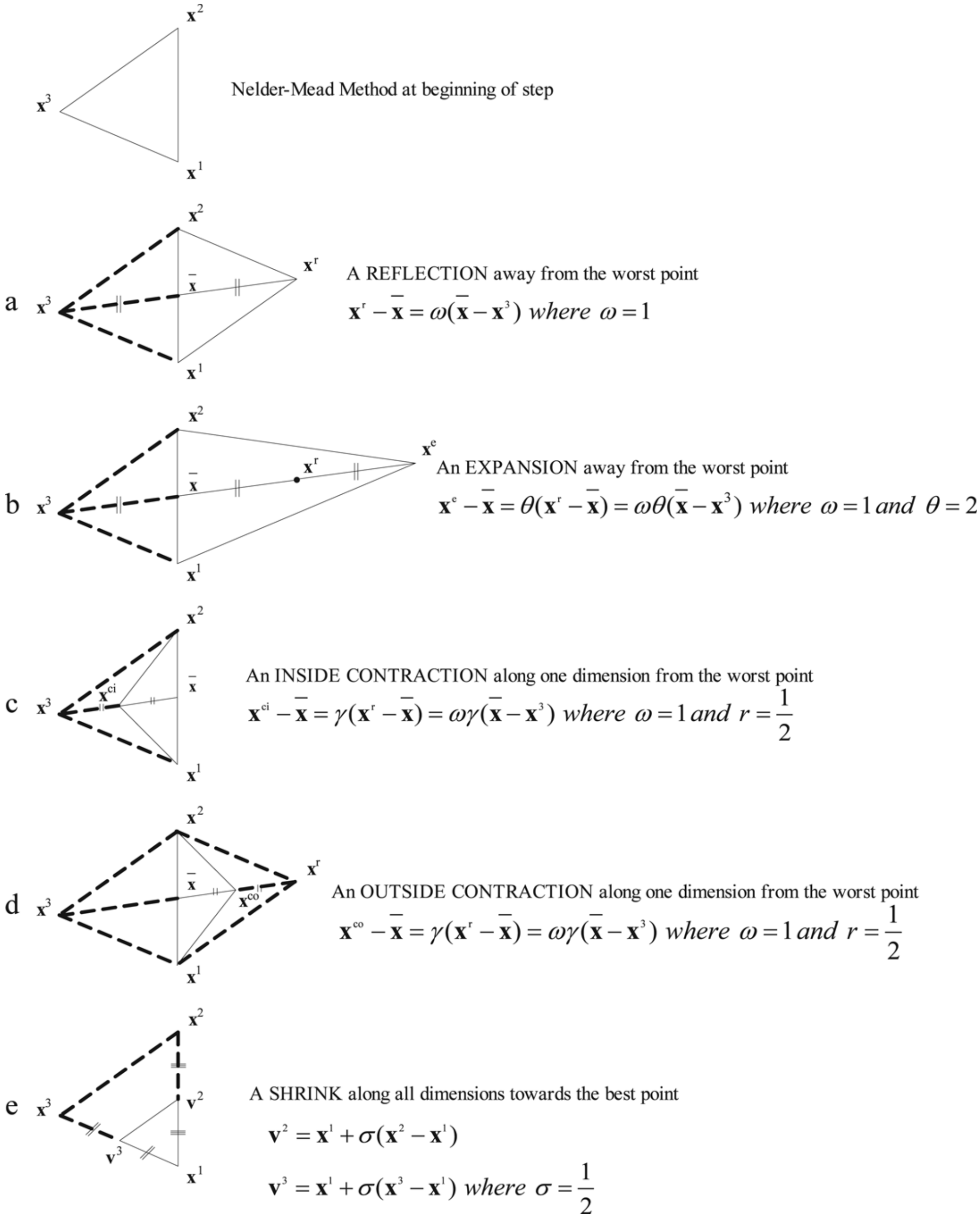

Fig. 2. Possible outcomes for a step in the Nelder–Mead simplex algorithm for $n = 2$.

horizon n is given. Next, we will show how to determine the optimal number of replenishments in the planning horizon n^*. Then, a complete search procedure to find the optimal number of replenishments n^* and the optimal replenishment time points t_i^* and shortage time points s_i^* will be presented.

4.3.1. *Finding the optimal replenishment time points t_i^* and s_i^**

Given the number of replenishments in the planning horizon, we need to identify initial vertices so that we can start performing the Nelder–Mead algorithm to find the optimal replenishment time point t_i^*. Since the number of replenishments is n and $t_0 = 0$, $t_n = H$, we need to find $(t_1, t_2, \ldots, t_{n-1})$ such that the total relevant cost is minimized.

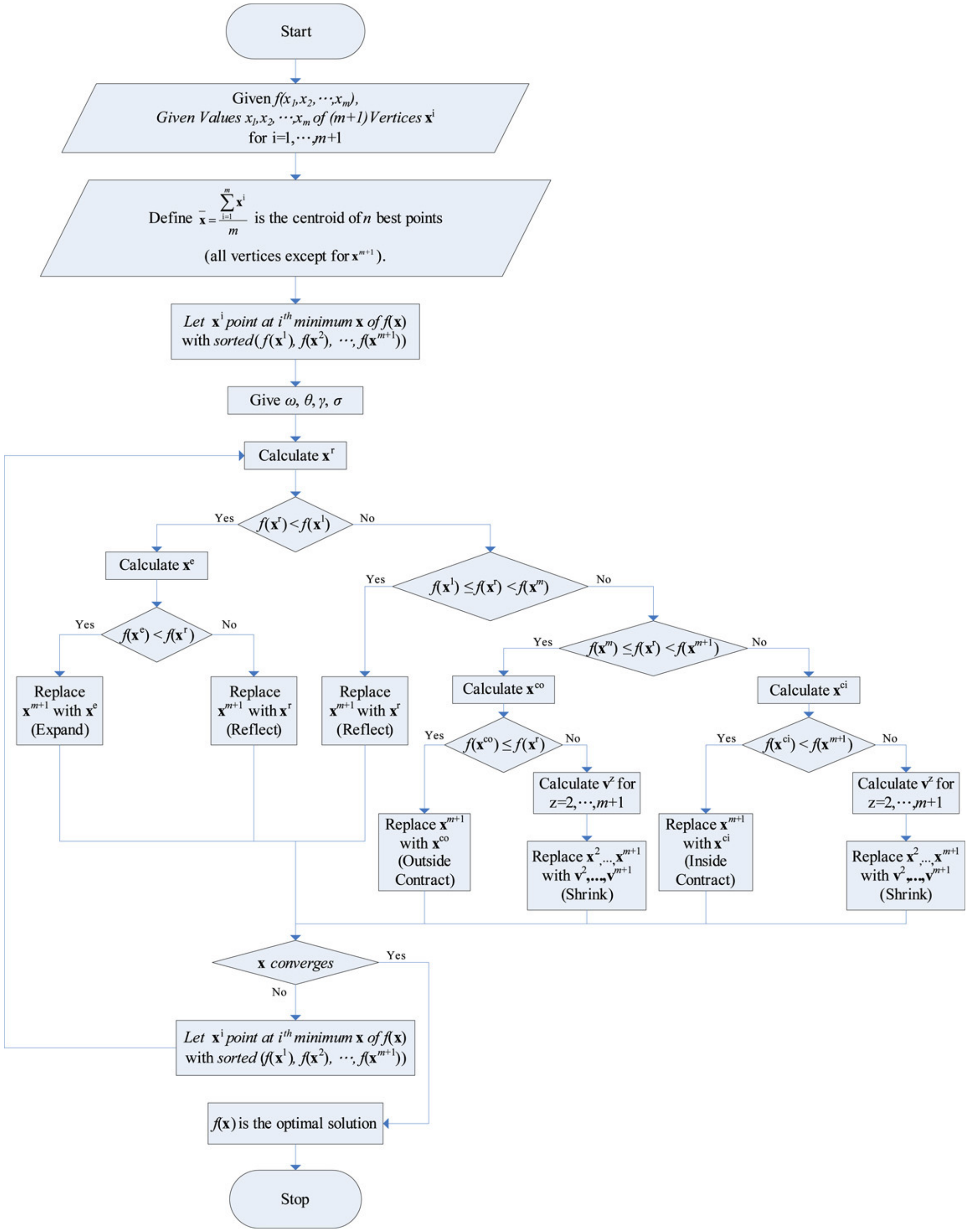

Fig. 3. Flow chart of iterations in the Nelder–Mead method.

Because we have $(n - 1)$ variables, we need to have n vertices initially. The first vertex can be obtained by dividing H equally. Namely,

$$\mathbf{t}^1 = \left(\frac{H}{n}, \frac{2H}{n}, \ldots, \frac{(n-1)H}{n}\right). \tag{9}$$

The second vertex is identical to $\mathbf{t}^1$ in Eq. (9) except that the first term is multiplied by a factor $(1 + \Delta)$. Similarly, the third vertex is identical to $\mathbf{t}^1$ in Eq. (9) except that the second term is multiplied by a factor $(1 + \Delta)$. The same operations can be performed to obtain the initial n vertices, which are shown in expression (11). We note that Δ is usually set to be 0.05.

$$\begin{bmatrix} \mathbf{t}^1 \\ \mathbf{t}^2 \\ \vdots \\ \mathbf{t}^{n-1} \\ \mathbf{t}^n \end{bmatrix} = \begin{bmatrix} \frac{H}{n} & \frac{2H}{n} & \cdots & \frac{(n-2)H}{n} & \frac{(n-1)H}{n} \\ \frac{(1+\Delta)H}{n} & \frac{2H}{n} & \cdots & \frac{(n-2)H}{n} & \frac{(n-1)H}{n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{H}{n} & \frac{2H}{n} & \cdots & \frac{(1+\Delta)(n-2)H}{n} & \frac{(n-1)H}{n} \\ \frac{H}{n} & \frac{2H}{n} & \cdots & \frac{(n-2)H}{n} & \frac{(1+\Delta)(n-1)H}{n} \end{bmatrix}. \tag{10}$$

After obtaining initial vertices $\mathbf{t}^1, \mathbf{t}^2, \ldots, \mathbf{t}^{n-1}, \mathbf{t}^n$, we can perform the Nelder–Mead algorithm for our problem. The searching procedure to find the optimal $(t_1^*, t_2^*, \ldots, t_{n-1}^*)$ is summarized as follows.

Step 0: Given the initial vertices $\mathbf{t}^i$ for $i = 1, \ldots, n$, and the relevant parameter $\omega, \theta, \gamma, \sigma$.

Step 1: Order the vertices $W(\mathbf{t}^1), W(\mathbf{t}^2), \ldots, W(\mathbf{t}^n)$, and let $\mathbf{t}^i$ to be the ith minimum $\mathbf{t}$ of $W(\mathbf{t})$ with sorted $(W(\mathbf{t}^1), W(\mathbf{t}^2), \ldots, W(\mathbf{t}^n))$.

Step 2: Compute the reflected vertex $\mathbf{t}^r$ from $\mathbf{t}^r = \bar{\mathbf{t}} + \omega(\bar{\mathbf{t}} - \mathbf{t}^n) = (1+\omega) \times \bar{\mathbf{t}} - \omega \times \mathbf{t}^n$ where $\bar{\mathbf{t}} = \frac{\sum_{i=1}^{n-1} \mathbf{t}^i}{n-1}$ is the centroid of the $n - 1$ best points (all vertices except for $\mathbf{t}^n$). And evaluate $W_r = W(\mathbf{t}^r)$.

Step 3: If $W_r < W(\mathbf{t}^1)$, go to step 4. If $W(\mathbf{t}^1) \le W_r < W(\mathbf{t}^{n-1})$, replace $\mathbf{t}^n$ with reflected vertex$\mathbf{t}^r$ and go to step 8. If $W(\mathbf{t}^{n-1}) \le W_r < W(\mathbf{t}^n)$, go to step 5. If $W(\mathbf{t}^n) \le W_r$, go to step 6.

Step 4: Compute the expansion vertex $\mathbf{t}^e$ from $\mathbf{t}^e = \bar{\mathbf{t}} + \theta(\mathbf{t}^r - \bar{\mathbf{t}}) = \bar{\mathbf{t}} + \omega\theta(\bar{\mathbf{t}} - \mathbf{t}^n) = (1+\omega\theta) \times \bar{\mathbf{t}} - \omega\theta \times \mathbf{t}^n$,and evaluate $W_e = W(\mathbf{t}^e)$. If $W_e < W_r$, replace $\mathbf{t}^n$ with expansion vertex $\mathbf{t}^e$ and go to step 8; otherwise, replace $\mathbf{t}^n$ with reflected vertex $\mathbf{t}^r$ and go to step 8.

Step 5: Compute the outside contraction vertex $\mathbf{t}^{\text{co}}$ from $\mathbf{t}^{\text{co}} = \bar{\mathbf{t}} + \gamma(\mathbf{t}^r - \bar{\mathbf{t}}) = (1+\omega\gamma) \times \bar{\mathbf{t}} - \omega\gamma \times \mathbf{t}^n$, and evaluate $W_{\text{co}} = W(\mathbf{t}^{\text{co}})$.

If $W_{\text{co}} \le W_r$, replace $\mathbf{t}^n$ with the outside contraction vertex $\mathbf{t}^{\text{co}}$ and go to step 8; otherwise, go to step 7.

Step 6: Compute the inside contraction vertex $\mathbf{t}^{\text{ci}}$ from $\mathbf{t}^{\text{ci}} = \bar{\mathbf{t}} - \gamma(\bar{\mathbf{t}} - \mathbf{t}^n) = (1-\gamma) \times \bar{\mathbf{t}} + \gamma \times \mathbf{t}^n$, and evaluate $W_{\text{ci}} = W(\mathbf{t}^{\text{ci}})$.

If $W_{\text{ci}} < W(\mathbf{t}^n)$, replace $\mathbf{t}^n$ with the inside contraction vertex $\mathbf{t}^{\text{ci}}$ and go to step 8; otherwise, go to step 7.

Step 7: Calculate the n vertices $\mathbf{v}^z = \mathbf{t}^1 + \sigma(\mathbf{t}^z - \mathbf{t}^1)$, for $z = 2, \ldots, n$.

Replace vertices $\mathbf{t}^2, \ldots, \mathbf{t}^n$ with vertices $\mathbf{v}^2, \ldots, \mathbf{v}^n$. Go to step 8.

Step 8: Order the vertices $W(\mathbf{t}^1), W(\mathbf{t}^2), \ldots, W(\mathbf{t}^n)$, and let $\mathbf{t}^i$ to be the ith minimum $\mathbf{t}$ of $W(\mathbf{t})$ with sorted $(W(\mathbf{t}^1), W(\mathbf{t}^2), \ldots, W(\mathbf{t}^n))$. Check if $\mathbf{t}$ converges; if yes, then output t_i^*, s_i^* (from Eq. (5)) and $W(t_i^*)$; otherwise, go to Step 2.

4.3.2. Finding the optimal number of replenishments n^*

Given n, from Section 4.3.1 we can obtain optimal t_i^* and s_i^*. For notational simplicity, we let

$$W(n) = W(n, \{t_i^*\}, \{s_i^*\}). \tag{11}$$

From Hariga [12], Teng [10] and Yang et al. [18], we note that the total relevant cost $W(n)$ is a strictly convex function of the number of replenishments n. If we assume that demand is uniformly distributed in the planning horizon, from EOQ formula, we can obtain the following result.

$$EOQ = \sqrt{\frac{2C_1 Q}{C_2 H}}. \tag{12}$$

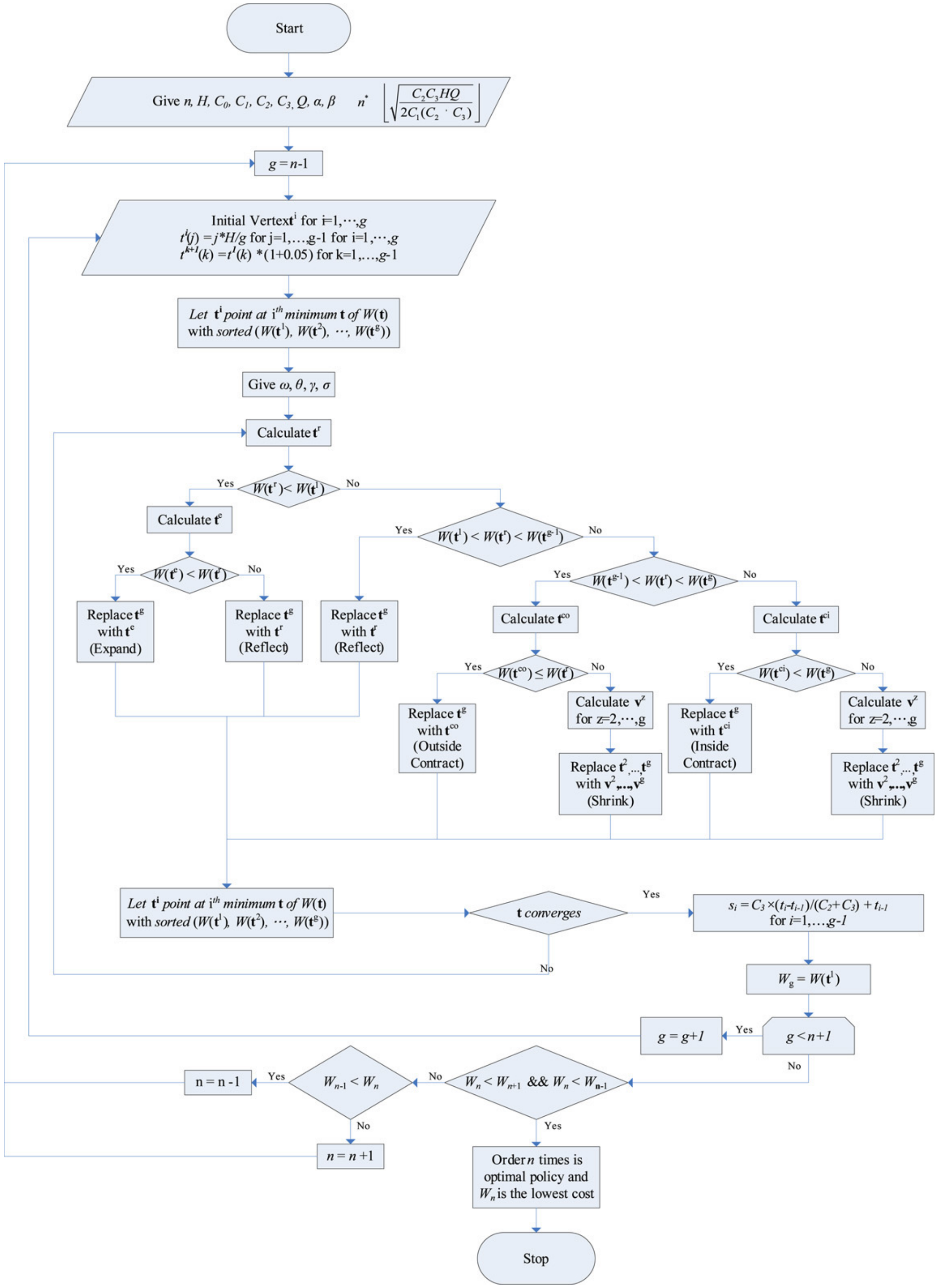

Fig. 4. Flow chart of the complete search procedure to find the optimal replenishment policy.

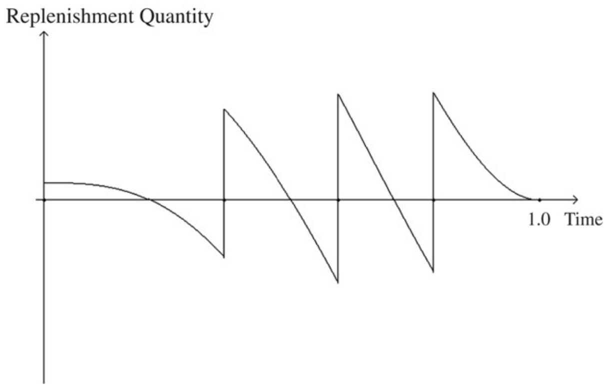

Fig. 5. Graphical representation of the inventory behavior.

Therefore, the approximate integer value for n can be expressed as Eq. (13).

$$n = \left\lfloor \sqrt{\frac{C_2 C_3 H Q}{2C_1(C_2 + C_3)}} \right\rfloor \tag{13}$$

where $\lfloor x \rfloor$ is the rounded integer of x. We note that Eq. (13) is identical to Eq. (18) in Yang et al. [18]. It is apparent that searching n^* for by starting with n as in Eq. (13) is much more efficient than starting with $n = 1$. Therefore, we can have the following procedure to find the optimal number of replenishments.

Step 0: Choose two initial values of n, say n in Eq. (13) and $n - 1$. Utilize the Nelder–Mead algorithm described in Section 4.3.1 to find $\{t_i^*\}$ and $\{s_i^*\}$ and compute $W(n)$ and $W(n - 1)$.

Step 1: If $W(n) \geq W(n - 1)$, then compute $W(n - 2)$, $W(n - 3)$ until we find $W(k) \leq W(k - 1)$. Output $n^* = k$ and stop.

Step 2: If $W(n) < W(n - 1)$, then compute $W(n + 1)$, $W(n + 2)$ until we find $W(k) < W(k + 1)$. Output $n^* = k$ and stop.

4.3.3. Combining the two methods to solve our problem

By combining the algorithms shown in Sections 4.3.1 and 4.3.2, we can have the complete search procedure to determine the optimal number of replenishments in the planning horizon n^* as well as the corresponding optimal replenishment time points t_i^* and the beginning time points of shortage s_i^*. The flow chart of the complete search procedure is shown in Fig. 4.

5. Numerical examples and sensitive analyses

In this section, we provide numerical examples to illustrate the features of the proposed model and the corresponding solution procedure. The values of parameters are $H = 1$, $C_0 = 10$, $C_1 = 50$, $C_2 = 5$, $C_3 = 7$, $Q = 5000$, $\alpha = 3$, and $\beta = 2$. By employing the algorithm presented in the previous sections, we can obtain the optimal solution which are $n^* = 4$, $\{t_i^*\} = \{0.0, 0.3643, 0.5945, 0.7876\}$, $\{s_i^*\} = \{0.2125, 0.4986, 0.7072\}$, and the corresponding total cost is 50 349.30. The graphical representation of the inventory behavior of this example can be shown as Fig. 5.

If we vary the shortage cost C_3 from 1 to 4, then the optimal solutions can be obtained and shown in Table 1.

From Table 1, we can have the following observations:

1. If the shortage cost C_3 increases, then the optimal n^* remains the same or increases.
2. Given $n^* = 3$, the optimal t_1^* increases from 0.3001 to 0.3141, and the optimal t_2^* decreases from 0.7997 to 0.7655 as C_3 changes from 1 to 2. On the other hand, given $n^* = 3$, the optimal s_1^* increases from 0.05 to 0.0898, and the optimal s_2^* increases from 0.3834 to 0.4431 as C_3 changes from 1 to 2.

Table 1
Sensitivity analysis for shortage cost C_3

Relationship of shortage cost and replenishment number				
C_3	1	2	3	4
n	3	3	4	4
t_0^*	0.0	0.0	0.0	0.0
t_1^*	0.3001	0.3141	0.2991	0.3806
t_2^*	0.7997	0.7655	0.5861	0.5742
t_3^*	–	–	0.8418	0.8189
s_1^*	0.05	0.898	0.1122	0.1691
s_2^*	0.3834	0.4431	0.4067	0.4666
s_3^*	–	–	0.6820	0.6829
W^*	50 240.85	50 287.07	50 306.11	50 317.35

Table 2
Sensitivity analysis for setup cost C_1

Relationship of replenishment cost and replenishment number							
C_1	20	30	40	50	60	70	80
n	6	5	4	4	4	4	3
t_0^*	0.0	0.0	0.0	0.0	0.0	0.0	0.0
t_1^*	0.2702	0.3024	0.3606	0.3606	0.3606	0.3606	0.3775
t_2^*	0.4454	0.4893	0.5787	0.5787	0.5787	0.5787	0.6886
t_3^*	0.5981	0.6657	0.7832	0.7832	0.7832	0.7832	–
t_4^*	0.7211	0.8102	–	–	–	–	–
t_5^*	0.8212	–	–	–	–	–	–
s_1^*	0.1801	0.2016	0.2404	0.2404	0.2404	0.2404	0.2517
s_2^*	0.3870	0.4270	0.5060	0.5060	0.5060	0.5060	0.5849
s_3^*	0.5472	0.6069	0.7150	0.7150	0.7150	0.7150	–
s_4^*	0.6801	0.7620	–	–	–	–	–
s_5^*	0.7878	–	–	–	–	–	–
W^*	50 229.70	50 282.32	50 329.25	50 369.25	50 409.25	50 449.25	50 482.17

3. Given $n^* = 4$, the optimal t_1^* increases from 0.2991 to 0.3806, the optimal t_2^* decreases from 0.5861 to 0.5742, and the optimal t_3^* decreases from 0.8418 to 0.8189 as C_3 changes from 3 to 4. On the other hand, given $n^* = 4$, the optimal s_1^* increases from 0.1122 to 0.1691, the optimal s_2^* increases from 0.4067 to 0.4666, and the optimal s_3^* increases from 0.6820 to 0.6829 as C_3 changes from 3 to 4.
4. The total cost W increases as the shortage cost C_3 increases.

Given values of the parameters are $H = 1$, $C_0 = 10$, $C_2 = 5$, $C_3 = 10$, $Q = 5000$, $\alpha = 3$, and $\beta = 2$. If we vary setup cost C_1 from 20 to 80, then the optimal solutions can be obtained and shown as in Table 2.

From Table 2, we can have the following observations:

1. If the setup cost C_1 increases, then the optimal n^* remains the same or decreases.
2. Given $n^* = 4$, different values C_1 will result in identical optimal t_i^* and s_i^*. This is because the setup cost is irrelevant in determining the optimal t_i^* and s_i^*. However, the change of the setup cost C_1 will affect the total relevant cost.
3. The total cost W increases as the setup cost C_1 increases.

Given values of the parameters are $H = 1$, $C_0 = 20$, $C_1 = 150$, $C_3 = 20$, $Q = 5000$, $\alpha = 3$, and $\beta = 2$. If we vary the inventory holding cost C_2 from 10 to 40, then the optimal solutions can be obtained and shown as in Table 3.

Table 3
Sensitivity analysis for setup cost C_2

Relationship of holding cost and replenishment number				
C_2	10	20	30	40
n	3	4	4	5
t_0^*	0.0	0.0	0.0	0.0
t_1^*	0.3775	0.2875	0.3184	0.1878
t_2^*	0.6886	0.5611	0.6052	0.4612
t_3^*	–	0.8069	0.8231	0.6519
t_4^*	–	–	–	0.8484
s_1^*	0.2517	0.1437	0.1274	0.0626
s_2^*	0.5849	0.4243	0.4331	0.2790
s_3^*	–	0.6837	0.6924	0.5248
s_4^*	–	–	–	0.7174
W^*	100 934.35	101 150.31	101 258.12	101 366.81

From Table 3, we can have the following observations:

1. If the inventory holding cost C_2 increases, then the optimal n^* remains the same or increases.
2. Given $n^* = 4$, the optimal t_1^* increases from 0.2875 to 0.3184, the optimal t_2^* increases from 0.5611 to 0.6052, and the optimal t_3^* increases from 0.8069 to 0.8231 as C_2 changes from 20 to 30. On the other hand, given $n^* = 4$, the optimal s_1^* decreases from 0.1437 to 0.1274, the optimal s_2^* increases from 0.4243 to 0.4331, and the optimal s_3^* increases from 0.6837 to 0.6924 as C_2 changes from 20 to 30.
3. The total cost W increases as the inventory holding cost C_2 increases.

6. Concluding remarks

In this paper, we explored the idea that the demand function follows the product-life-cycle shape for the decision maker in order to determine the optimal number of replenishments, the corresponding optimal replenishment time points, and the shortage beginning time points in the planning horizon. A complete search procedure is provided to find the optimal solution by employing the Nelder–Mead algorithm.

Possible extensions of the approach proposed in this paper may be made by considering: (1) the deteriorating nature of the product; (2) the discounting effects due to time value of money; (3) quantity discounts of the unit cost, . . ., and so on.

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References

[1] L.B. Schwarz, Economic order quantities for products with finite demand horizons, AIIE Transactions 4 (3) (1972) 234–237.
[2] W.A. Donaldson, Inventory replenishment policy for a linear trend in demand — An analytical solution, Operational Research Quarterly 28 (3) (1977) 663–670.
[3] E.A. Silver, A simple inventory replenishment decision rule for a linear trend in demand, Journal of the Operational Research 30 (1) (1979) 71–75.
[4] R.J. Henery, Inventory replenishment policy for increasing demand, Journal of the Operational Research Society 30 (7) (1979) 611–617.
[5] E. Ritchie, The E.O.Q. for linear increasing demand: A simple optimal solution, Journal of the Operational Research 35 (10) (1984) 949–953.
[6] M. Amrani, G.K. Rand, An eclectic algorithm for inventory replenishment for items with increasing linear trend in demand, Engineering Costs and Production Economics 19 (1990) 261–266.
[7] J. Yang, G.K. Rand, An analytic eclectic heuristic for replenishment with linear increasing demand, International Journal of Production Economics 32 (1993) 261–266.
[8] J.T. Teng, A note on inventory replenishment policy for increasing demand, Journal of the Operational Research 45 (11) (1994) 1335–1337.

[9] S.K. Goyal, Determining economic replenishment intervals for linear trend in demand, International Journal of Production Economics 34 (1994) 115–117.
[10] J.T. Teng, A deterministic inventory replenishment model with a linear trend in demand, Operations Research Letters 19 (1996) 33–41.
[11] L.J.A. Brosseau, An inventory replenishment policy for the case of a linear decreasing trend in demand, INFOR 20 (3) (1982) 252–257.
[12] M. Hariga, Comparison of heuristic procedures for the inventory replenishment problem with a linear trend in demand, Computers & Industrial Engineering 28 (2) (1995) 245–258.
[13] W.Y. Lo, C.H. Tsai, R.K. Li, Exact solution of inventory replenishment policy for a linear trend in demand — Two equation model, International Journal of Production Economics 76 (2002) 111–120.
[14] G.Q. Zhao, J. Yang, G.K. Rand, Heuristics for replenishment with linear decreasing demand, International Journal of Production Economics 69 (2001) 339–345.
[15] S.K. Goyal, B.C. Giri, A simple rule for determining replenishment intervals of an inventory with linear decreasing demand rate, International Journal of Production Economics 83 (2003) 139–142.
[16] J. Yang, G.Q. Zhao, G.K. Rand, Comparison of several heuristics using an analytical procedure for replenishment with nonlinear increasing demand, International Journal of Production Economics 58 (1999) 49–55.
[17] S.P. Wang, On inventory replenishment with non-linear increasing demand, Computers & Operations Research 29 (2002) 1819–1825.
[18] H.L. Yang, J.T. Teng, M.S. Chern, A forward recursive algorithm for inventory lot-size models with power-form demand and shortages, European Journal of Operational Research 137 (2002) 394–400.
[19] M.H. Roger, Batching policies for a product life cycle, International Journal of Production Economics 45 (1996) 421–427.
[20] L. Benkherouf, On the replenishment policy for an inventory model with linear trend in demand and shortages, Journal of the Operational Research Society 45 (1) (1994) 121–122.
[21] U. Dave, A deterministic lot-size inventory model with shortages and a linear trend in demand, Naval Research Logistics 36 (1989) 507–514.
[22] U. Dave, On a heuristic inventory-replenishment rule for items with a linearly increasing demand incorporating shortages, Journal of Operational Research Society 40 (9) (1989) 827–830.
[23] M. Deb, K. Chaudhuri, A note on the heuristic for replenishment of trended inventories considering shortages, Journal of Operational Research Society 38 (5) (1987) 459–463.
[24] S.K. Goyal, A heuristic for replenishment of trended inventories considering shortages, Journal of Operational Research Society 39 (9) (1988) 885–887.
[25] M. Hariga, The Inventory lot-sizing problem with continuous time-varying demand and shortages, Journal of Operational Research Society 45 (7) (1994) 827–837.
[26] T.M. Murdeshwar, Inventory replenishment policy for linearly increasing demand considering shortages — An optimal solution, Journal of Operational Research Society 39 (7) (1988) 687–692.
[27] J.T. Teng, M.S. Chern, H.L. Yang, An optimal recursive method for various inventory replenishment models with increasing demand and shortages, Naval Research Logistics 44 (1997) 791–806.
[28] J.A. Nelder, R. Mead, A simplex method for function minimization, Computer Journal 7 (1965) 308–313.
[29] J.C. Lagarias, J.A. Reeds, M.H. Wright, P.E. Wright, Convergence properties of the Nelder–Mead simplex method in low dimensions, SIAM Journal of Optimization 9 (1) (1998) 112–147.